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FIELD THEORETICAL APPROACH TO QUANTUM HALL FERROMAGNETS

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Abstract

We present a quantum field theoretical analysis of a $\nu = 1$ quantum Hall system when the effective Landé g factor is small. We clearly demonstrate that the ground state of the system is ferromagnetic. We note that it is the short range component of the Coulomb interaction which is instrumental in aligning the spins. We then go on to derive the effective lagrangian for the lowest lying spin excitations. At the leading order, apart from the usual O(3) sigma model terms, we find a term proportional to the topological Pontryagin density and a long range Coulomb interaction term between these densities. Beyond the leading order in the derivative expansion, we find an interesting Chern-Simons term constructed out of the basic spin variables. For low enough energies, however, we notice that the effects

of mixing of higher Landau levels is more important than the next to leading terms in the derivative expansion. We provide a systematic way of calculating these corrections.

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1. Introduction

Over the last few years, a lot of interest has been generated over the physics of the quantum Hall effect of multicomponent systems [1]. These components could be electron spin, electrons in different valleys (as in Si systems) or even be electrons in different layers in multi-layered samples.

In this article, we shall focus on the case of a single-layer system, where, owing to various circumstances, the electron spin becomes a relevant SU(2) degree of freedom.

Specifically, we shall be discussing systems containing fermions living on a plane, interacting via the Coulomb interaction and subjected to a strong magnetic field orthogonal to the plane. These non-relativistic electrons couple minimally as well as through the Pauli term to the magnetic field. In the absence of the Coulomb repulsion, the single-particle spectrum is well-known: It comprises of equi-spaced Landau levels, each with an infinite degeneracy. If the Landé g factor of the electrons retains its vacuum value, namely, $g = 2$, the Zeeman split between opposite spins is precisely equal to the gap between any two successive Landau levels. Therefore, if only the lowest Landau level (L.L.L.) is filled, the system is automatically spin polarised. In many realistic systems(e.g. GaAs), however, band effects enhance the Landau level(L.L.) separation by a factor of ~ 20 and reduce the Zeeman splitting by ~ 4 [2]. For typical experimental situations, the L.L. separation, in temperature units, is $\sim 200K$, the Coulomb interaction is $\sim 100K$ and the Zeeman split is $\sim 2K$. In view of these numbers, it is clear that the effective g factor is $\ll 2$. In this case, the gap between successive single-particle levels is reduced considerably and the spin degree of freedom can be expected to play a signifant role in these systems.

Also, from the numbers quoted above, it would seem that the relevant gap in these systems would be the Zeeman gap rather than the Landau gap. However, in the case of odd-integer fillings, experimentally measured gaps are seen to exceed the single-particle Zeeman gap by a factor of as much as 20 [3]. Thus it seems that for these odd-integral

Hall systems, the interaction between the electrons plays a decisive role in generating the observed gap. In these cases, the conventional distinction that is made between the integer and the fractional Hall effects gets blurred.

Here, we consider the simplest odd-integral Hall system with the filling fraction $\nu = 1$. We find that the incompressible ground state of the system is spin-polarised. This is true even when g is set equal to zero, when the underlying microscopic action acquires an exact SU(2) spin symmetry. Thus the ground state breaks this symmetry down to a U(1) phase symmetry. This is the hallmark of spontaneous symmetry breaking (SSB) and thus the ground state is ferromagnetic. The lowest-lying excitations are off course the neutral gapless magnons which are associated with the broken generators of the spin SU(2). These are long wavelength spin waves. However, in these quantum Hall ferromagnets, there are, in contradistinction with usual ferromagnets, low-lying charged excitations of macroscopic spin and nontrivial spin order. These are actually the dominant charged excitations whenever $g \leq g_{cr}$ and when ν deviates slightly from 1 [3]. They are known as spin-skyrmions and are topological in nature, their topological charge (Pontryagin number) equalling their electric charge. Under the conditions just mentioned, they are actually more relevant than charged single-particle excitations. Experiments [4] seem to provide support for the existence of these topological excitations.

The system has already been studied in a variety of ways [3,5,6,7,8]. These works all convey the essential physics quite accurately. However, it seems to us that there are lacunae in the actual derivation of the effective action for the spin starting from a microscopic theory, as we argue below.

In [3,6], the effective action has been motivated rather than derived from the microscopics. Thus some terms that do not occur in usual ferromagnetic sigma-model effective actions have not been discussed. The first-quantised approach espoused in [5] is predicated to discussing only the lowest Landau level (L.L.L.) and it is not immediately obvious how the effects of L.L. mixing by the Coulomb term may be incorporated. Furthermore, the

distinctness of the rather different roles played by the long and the short distance components of the Coulomb interaction in the derivation of the effective action has not, in our opinion, been suitably emphasized. Namely, that it is the short distance component that provides the magnon kinetic energy and the long distance part that mediates a Coulomb repulsion between skyrmions has not been spelled out clearly enough. The interaction that leads to a ferromagnetic ground state is the Coulomb interaction which is somewhat counter-intuitive as it is a nonlocal interaction. Thus it is in our opinion vastly clarifying to see ferromagnetism emerge from the short distance part of this nonlocal interaction. In [8], a Hartree-Fock theory of skyrmions has been proposed, but the contribution to the effective spin action due to Landau Level mixing has not been addressed.

The existence of SSB and the dynamics of the resulting Goldstone modes is studied most easily within a second-quantised (quantum field theoretic) framework. A further clear advantage of a field theoretic formulation is that effects due to mixing with the higher L.L. may be systematically accounted for. In this article, therefore, we have formulated our problem in a second-quantised language.

Thus, the system is taken to be governed by a second-quantised fermionic action whose explicit form will be given in the next section. For it to desribe a $\nu = 1$ Hall system, the corresponding ground state must be made up of only the L.L.L. single-particle states. Therefore we make a very general ansatz for the density matrix describing such a ground state, where this matrix is constructed out of only the L.L.L. single-particle states. We then compute the mean energy of the ground state using this ansatz and the second-quantised Hamiltonian for the system. The requirement that this energy should be minimised automatically fixes the parameters in the ansatz such that the matrix describes a ground state with uniform density and magnetisation. We see that the energy is a minimum when all the spins are aligned in any given direction, all choices of this direction being equivalent. Thus we choose a given direction and thereby break the SU(2) symmetry of the action spontaneously by our choice of the ground state.

The description of the Goldstone excitations above this ground state is the clearest in the nonlinear realisation [9] of spontaneous symmetry breaking. Namely, if a continuous symmetry group G is broken spontaneously to a group H, where H is a proper subgroup of G, the Goldstone modes are described by a slowly-varying unitary matrix U living in the coset space G/H. U transforms nonlinearly under transformations g belonging to G. The density matrix itself, by construction should transform covariantly under $g \in G$. In our case we generalise the ansatz to include the magnons by introducing an appropriate unitary matrix U and requiring that the resultant density matrix be covariant under SU(2), *projected onto the L.L.L.*. This last requirement entails the inclusion of appropriate derivatives of U to render the matrix covariant.

We further use this density matrix to compute the mean action, which is now expressed in terms of the unitary matrix U and is the effective action for the spin degrees of freedom. A rather interesting feature of L.L.L. projection is that charge and spin densities do not commute. This leads to the possibility that some non-trivial spin configurations may be charged, which is realised through the spin-skyrmions.

The effective action thus obtained contains all the usual terms one expects for the ferromagnetic magnon system. Namely, we get a Wess-Zumino term with a single time-derivative and a standard magnon kinetic energy term that is quadratic in spatial derivatives. These ensure that the magnon dispersion is quadratic, as is the case for ferromagnets. Furthermore, we get the Coulomb repulsion between the topological charge densities first obtained in [3]. Over and above these terms, however, we get some terms which have been noticed only recently [10]. There is a term proportional to the Pontryagin number which implies that topological excitations are energetically favoured. Further, there is a Chern-Simons term that can be constructed out of U which may result in the statistical transmutation of the quasiparticles [11]. These emerge with a minimum of fuss in our formalism of nonlinear realisation of the spin SU(2).

As mentioned above, another advantage of a field theoretic formulation is the ease

with which contributions due to mixing with higher L.L. are computed. Here, we envision a path integral approach, whereby the higher L.L. may be integrated out explicitly to yield an effective action for the L.L.L. As an example we show how the magnon kinetic term and the topological term acquire finite renormalisation due to this mixing.

The organisation of the article is as follows. In section 2, we establish the notation and describe the projection onto the L.L.L. In section 3, we show that an ansatz for the density matrix for the ground state involving only the L.L.L. automatically yields a ferromagnetic ground state when energy minimisation is required. In section 4, we generalise the ansatz to include the Goldstone modes and obtain the effective action for the spin degree of freedom. Leading corrections due to mixing with higher L.L. are obtained in section 5. We conclude in section 6 with a critical evaluation of the contents of this article and a discussion of the avenues for further investigations. Details of the calculation and a brief discussion of nonlinear realisations of symmetries have been relegated to the appendices.

2. Notation and Formulation

In this section, we shall write down the microscopic action for the system at hand, solve the single-particle problem in the absence of the Coulomb interaction and establish the notation. As far as the basic notation is concerned, we follow [12] as closely as possible.

We also discuss the projection to the L.L.L. and write down the form of the Coulomb term after projection on to the L.L.L.

The microscopic action for nonrelativistic planar spinning fermions in a magnetic field perpendicular to the plane is:

$$S = S_0 + S_c \tag{2.1}$$

where

$$S_0 \equiv \int dt \int d\vec{x}\ \hat{\psi}^\dagger(\vec{x},t)\big[i\partial_t - \frac{1}{2m}(\vec{p}-\vec{A})^2 + \frac{gB}{2m}S_z + \mu\big]\hat{\psi}(\vec{x},t) \tag{2.2}$$

Here, we have rescaled $e\vec{A} \to \vec{A}$ and thus, e does not appear explicitly in (2.2). Also,

$$S_c \equiv -\frac{1}{2}\int dt \int d\vec{x} \int d\vec{x}' \ \hat{\psi}^\dagger(\vec{x},t)\hat{\psi}^\dagger(\vec{x}',t)V(|\vec{x}-\vec{x}'|)\hat{\psi}(\vec{x}',t)\hat{\psi}(\vec{x},t) \qquad (2.3).$$

Here, m is the effective mass, g the effective g factor and μ the chemical potential that specifies the particle content of the system. We are going to take $V(|\vec{x}-\vec{x}'|) = \frac{e^2}{|\vec{x}-\vec{x}'|}$, which is the Coulomb repulsion. Here, we have retained e explicitly, as, in the sequel, this will be seen to generate a new scale in the problem.

$\hat{\psi}_\alpha$ is a 2-component fermion operator obeying $\{\hat{\psi}_\alpha(\vec{x}), \hat{\psi}^\dagger_\beta(\vec{x}')\} = \delta_{\alpha\beta}\delta(\vec{x}-\vec{x}')$.

Further, let

$$h_0 \equiv \frac{1}{2m}(\vec{p}-\vec{A})^2 - \frac{gB}{2m}S_z \qquad (2.4)$$

and

$$\vec{\nabla} \wedge \vec{A} = -B, \qquad (2.5)$$

where B is the applied strong magnetic field.

The single-particle spectrum of h_0 is readily obtained.

Let

$$\hat{\Pi}^x \equiv -i\partial_x - A^x$$

and

$$\hat{\Pi}^y \equiv -i\partial_y - A^y \qquad (2.5)$$

where

$$\vec{A} = (\alpha By, -(1-\alpha)Bx) \qquad (2.6),$$

α being some arbitrary parameter that interpolates between various gauge choices and drops out of physical quantities.

Further, let

$$\hat{\Pi} \equiv \hat{\Pi^x} - i\hat{\Pi^y}$$

and

$$\hat{\Pi^\dagger} \equiv \hat{\Pi^x} + i\hat{\Pi^y} \tag{2.7}.$$

Then,

$$h_0 = \frac{1}{2m}\hat{\Pi^\dagger}\hat{\Pi} + \frac{B}{2m}(1 - gS_z) \tag{2.8}.$$

The spectrum of h_0 is highly degenerate and this degeneracy is exposed by the "guiding centre" operator

$$\hat{X} \equiv \hat{x} - \frac{1}{B}\hat{\Pi^y} \tag{2.10}$$

which commutes with h_0.

The canonical conjugate of $\hat{X}$ is

$$\hat{Y} \equiv \hat{y} + \frac{1}{B}\hat{\Pi^x} \tag{2.11}.$$

We form the holomorphic and anti-holomorphic combinations of these:

$$\hat{Z} \equiv \hat{X} + i\hat{Y}$$

and

$$\hat{\bar{Z}} \equiv \hat{X} - i\hat{Y} \tag{2.12}.$$

Then,

$$\left[\hat{Z}, \hat{\bar{Z}}\right] = \frac{2}{B} \tag{2.13}.$$

Further,

$$\left[\hat{Z}, \hat{\Pi}\right] = \left[\hat{\bar{Z}}, \hat{\Pi}\right] = 0 \tag{2.14}.$$

We choose the eigenstates of h_0 to be $\left\{|n, \xi, \alpha\rangle\right\}$ where,

$$\begin{aligned}
\hat{\Pi}|n, \xi, \alpha\rangle &= \sqrt{2Bn}|n-1, \xi, \alpha\rangle \\
\hat{\Pi}^\dagger|n, \xi, \alpha\rangle &= \sqrt{2B(n+1)}|n+1, \xi, \alpha\rangle \\
\hat{Z}|n, \xi, \alpha\rangle &= \xi|n, \xi, \alpha\rangle \\
\hat{S}_z|n, \xi, \alpha\rangle &= \alpha|n, \xi, \alpha\rangle
\end{aligned} \tag{2.15}.$$

Here, $\alpha = \pm\frac{1}{2}$. Thus,

$$h_0|n, \xi, \alpha\rangle = E_{n,\alpha}|n, \xi, \alpha\rangle \tag{2.16}$$

where

$$E_{n,\alpha} = (n + \frac{1}{2} - \frac{g\alpha}{2})\omega_c \tag{2.17}.$$

Here, $\omega_c \equiv \frac{B}{m}$, the cyclotron frequency.

Therefore, for $g = 2$, the energy difference between $\alpha = \pm\frac{1}{2}$ is precisely ω_c, as mentioned in the introduction.

Now, $|n, \xi, \alpha\rangle = |n, \xi\rangle \otimes |\alpha\rangle$, where $|\xi\rangle$ is a coherent state of $\hat{Z}$.

Namely,

$$\hat{Z}|\xi\rangle = \xi|\xi\rangle \tag{2.18}.$$

Explicitly,

$$|\xi\rangle = \exp^{\frac{B}{2}\xi\hat{\bar{Z}}}|0\rangle \tag{2.19},$$

where $\hat{Z}|0\rangle = 0$.

The resolution of the identity in terms of these coherent states is:

$$I = \frac{B}{2\pi}\int d^2\xi \ \mathrm{e}^{-\frac{B}{2}|\xi|^2}|\xi\rangle\langle\bar{\xi}| \tag{2.20},$$

where,

$$d^2\xi \equiv d\ (\mathrm{Re}\xi)\mathrm{d}\ (\mathrm{Im}\xi)$$

.

The inner product of these coherent states is

$$\langle\bar{\eta}|\xi\rangle = \mathrm{e}^{\frac{B}{2}\bar{\eta}\xi} \tag{2.21}.$$

Now, the eigenfunctions are obtained by projecting $|n,\xi,\alpha\rangle$ onto coordinate space.

Specifically, the L.L.L. wavefunction is given by

$$\langle\vec{x}|0,\xi,\alpha\rangle = \sqrt{\frac{B}{2\pi}}\mathrm{e}^{-\frac{B}{4}|z|^2+\frac{B}{2}\bar{z}\xi} \tag{2.22},$$

where $z = x + iy$ and $\bar{z}$ is its complex conjugate.

The projector onto the L.L.L. is therefore given by

$$P_0 \equiv \frac{B}{2\pi}\int d^2\xi \ \mathrm{e}^{-\frac{B}{2}|\xi|^2}|0,\xi\rangle\langle 0,\bar{\xi}| \tag{2.23}.$$

Thus, from (2.22) and (2.23),

$$\langle\vec{x}|P_0 = \sqrt{\frac{B}{2\pi}}\mathrm{e}^{-\frac{B}{4}|z|^2}\langle 0,\bar{z}|P_0 \tag{2.24}$$

where $\bar{z}$, the coherent state parameter in (2.24), is actually constructed out of the spatial coordinates x and y. Namely, $\bar{z} = x - iy$.

Let us denote by $|\psi\rangle$ the second-quantised field operator such that

$$\hat{\psi}(\vec{x}, t) \equiv \langle \vec{x} | \psi(t) \rangle \tag{2.25}$$

Thus if we project the fields to the L.L.L., from (2.24) and (2.25), we have

$$\hat{\psi}(\vec{x}, t) \to \langle \vec{x} | P_0 | \psi(t) \rangle \equiv \hat{\psi}_0(\vec{x}, t) = \mathrm{e}^{-\frac{B}{4}|z|^2} \hat{\psi}_0(\bar{z}, t) \tag{2.26}$$

With this projector P_0, we can easily project the entire action to the L.L.L.:

$$\begin{aligned} S = \int dt \int d\vec{x} \, \big[\hat{\psi}_0^\dagger(\vec{x}, t) \{ i\partial_t - \frac{\omega_c}{2}(1 - gS_z) + \mu \} \hat{\psi}_0(\vec{x}, t) \\ - \frac{1}{2} \int d\vec{x}' \; \hat{\psi}_0^\dagger(\vec{x}, t) \hat{\psi}_0^\dagger(\vec{x}', t) V(|\vec{x} - \vec{x}'|) \hat{\psi}_0(\vec{x}', t) \hat{\psi}_0(\vec{x}, t) \big] \end{aligned} \tag{2.27}$$

Using (2.26), the Coulomb term in (2.27) is rewritten as

$$S_c \equiv -\frac{1}{2} \int dt \int d^2z d^2z' \mathrm{e}^{-\frac{B}{2}(|z|^2 + |z'|^2)} \hat{\psi}_0^\dagger(z, t) \hat{\psi}_0^\dagger(z', t) V(|z - z'|) \hat{\psi}_0(\bar{z'}, t) \hat{\psi}_0(\bar{z}, t) \tag{2.28}$$

where $d^2z \equiv \frac{B}{2\pi} d(\mathrm{Re\ z}) \mathrm{d}(\mathrm{Im\ z})$. We should further include a uniform neutralising background charge density in the system, equal in magnitude to the density associated with a completely filled L.L.L. Namely, $\rho_b = \frac{B}{2\pi}$. Including this explicitly, the Coulomb term is

$$S_c = -\frac{1}{2} \int dt \int d^2z d^2z' \big[\mathrm{e}^{-\frac{B}{2}|z|^2} \hat{\psi}_0^\dagger(z, t) \hat{\psi}_0(\bar{z}, t) - \frac{B}{2\pi} \big] V(|z - z'|) \big[\mathrm{e}^{-\frac{B}{2}|z'|^2} \hat{\psi}_0^\dagger(z', t) \hat{\psi}_0(\bar{z'}, t) - \frac{B}{2\pi} \big] \tag{2.29}$$

3. The Coulomb term and the ground state

Right at the onset, we should mention that in sections 3 and 4, we shall omit the suffix 0 from the fermionic fields, as we shall be concerned only with the L.L.L.

The Coulomb term projected onto the lowest Landau level reads (see appendix B)

$$\begin{aligned} L_c = &-\frac{1}{2}\int d^2z_1 d^2z_2 e^{-\frac{B}{2}|z_1^2|-\frac{B}{2}|z_2^2|}\left(\psi^\dagger(z_1)\psi(\bar{z}_1)-\rho_0(z_1,\bar{z}_1)\right) \\ &V(z_1-z_2,\bar{z}_1-\bar{z}_2)\left(\psi^\dagger(z_2)\psi(\bar{z}_2)-\rho_0(z_2,\bar{z}_2)\right) \end{aligned} \tag{3.1}$$

where the spin indices have been omitted, and the neutralizing background charge ρ_0 projected onto the lowest Landau level and the Coulomb potential V are given by

$$\rho_0(z,\bar{z}) = \frac{B}{2\pi}e^{\frac{B}{2}|z^2|}$$

$$V(z,\bar{z}) = \int \frac{d^2k}{2\pi}V(k,\bar{k})e^{\frac{i\bar{k}z+i\bar{z}k}{2}} \qquad , \quad V(k,\bar{k}) = \frac{e^2}{|k|} \tag{3.2}$$

Notice that the term above is most important at short distances. Let us then separate the short distance contribution from the long distance one by introducing an explicit cut-off λ, the size of which will be discussed later on

$$\begin{aligned} V(z,\bar{z}) =& V_l(z,\bar{z}) + V_s(z,\bar{z}) \\ V_l(z,\bar{z}) =& \int^\lambda \frac{d^2k}{2\pi}V(k,\bar{k})e^{\frac{i\bar{k}z+i\bar{z}k}{2}} \qquad , \quad V_s(z,\bar{z}) = \int_\lambda \frac{d^2k}{2\pi}V(k,\bar{k})\, e^{\frac{i\bar{k}z+i\bar{z}k}{2}} \end{aligned} \tag{3.3}$$

Let us assume first that only the short distance contribution is important in order to establish the properties of the ground state. We shall prove later on that this assumption is self-consistent. We have

$$L_{c,s} = -\frac{1}{2}\int d^2z_1 d^2z_2 e^{-\frac{B}{2}|z_1|^2-\frac{B}{2}|z_2|^2}\psi^\dagger(z_1)\psi(\bar{z}_1)V_s(z_1-z_2,\bar{z}_1-\bar{z}_2)\psi^\dagger(z_2)\psi(\bar{z}_2) \tag{3.4}$$

since the neutralising constant charge density gives no contribution for large (non-zero) momenta. From (3.4) it is not apparent at all that the spin degree of freedom is important to establish the nature of the ground state. Recall that we are interested in the situation where the density of electrons is equal to the density required to exactly fill up the lowest Landau level of one of the spin degrees of freedom (which dictates the choice of the neutralising background charge as given in (3.2)). However we cannot guarantee a priori that

all the electrons are going to be in the spin up or down state. As dicussed in the previous section this situation is suitably dealt with by the introduction of a density matrix .

$$\rho_{\alpha\beta}(\bar{z}, z') =< \bar{z}|\hat{\rho}_{\alpha\beta}|z' >= -\psi_\alpha(\bar{z})\psi^\dagger_\beta(z') \tag{3.5}$$

Notice that the mean local density in nothing but

$$\langle\rho(\vec{x})\rangle = tr(\rho(\bar{z}, z))e^{\frac{B}{2}|z|^2} \tag{3.6}$$

Let us take as an ansatz

$$\hat{\rho}_{\alpha\beta} = \frac{B}{2\pi}\int d^2\xi e^{-\frac{B}{2}|\xi|^2} M_{\alpha\beta}(\bar{\xi}, \xi)|\xi >< \bar{\xi}| \tag{3.7}$$

with $M_{\alpha\beta}(\bar{\xi}, \xi)$ being a slowly varying matrix-valued function. Recall that if $M_{\alpha\beta}(\bar{\xi}, \xi)$ is constant then the mean local density (3.6) is also constant. We have, using the results of Appendix A,

$$\rho_{\alpha\beta}(\bar{z}, z') \sim e^{-\frac{B}{2}\bar{z}z'}\left(M_{\alpha\beta}(\bar{z}, z') + \frac{2}{B}\partial_z\partial_{\bar{z}}M_{\alpha\beta}(\bar{z}, z') + \cdots\right) \tag{3.8}$$

For the Coulomb term (3.4) there are two inequivalent ways of introducing the density matrix $\hat{\rho}_{\alpha\beta}$ (3.7) , namely

$$\begin{aligned} &i) \qquad \psi^\dagger(z_1)\psi(\bar{z}_1)\psi^\dagger(z_2)\psi(\bar{z}_2) \longrightarrow \rho_{\alpha\alpha}(\bar{z}_1, z_1)\rho_{\beta\beta}(\bar{z}_2, z_2) \\ &ii) \qquad \psi^\dagger(z_1)\psi(\bar{z}_1)\psi^\dagger(z_2)\psi(\bar{z}_2) \longrightarrow -\rho_{\alpha\beta}(\bar{z}_1, z_2)\rho_{\beta\alpha}(\bar{z}_2, z_1) \end{aligned} \tag{3.9}$$

Consider first the case i). For constant $M_{\alpha\beta}(\bar{\xi}, \xi)$, which characterises constant space-time configurations, we obtain

$$L_{c,s} = 0 \tag{3.10}$$

This follows immediatly from (3.4) and (3.3). Indeed, upon integrating (3.4) over z_1 and $\bar{z}_1$ we obtain a delta function of k and $\bar{k}$ which, owing to the infrared cut-off for $V_s(z, \bar{z})$ in (3.3), gives zero. Consider next the case ii). Again, let us choose $M_{\alpha\beta}(\bar{z}, z')$ constant and take it as

$$M = \frac{\rho_0}{2} + \vec{m}\vec{\sigma} \tag{3.11}$$

We obtain

$$L_{c,s} = \frac{2\pi}{B}\int d^2z \int_\lambda \frac{d^2k}{2\pi} V(k,\bar{k}) e^{-\frac{\bar{k}k}{2B}} \left(\frac{\rho_0^2}{2} + 2\vec{m}^2\right) \tag{3.12}$$

where the role of the spin is immediate: the system wants to be with the largest possible constant magnetization $\vec{m}$ which gives rise to a negative contribution to the energy. This is only achieved by having all electrons with the spin pointing in the same direction, no matter whether the direction is up or down. Namely, we have a situation of spontaneous symmetry breaking and the ground state is ferromagnetic. We take, for definiteness,

$$\vec{m} = (0, 0, \frac{\rho_0}{2});\ \rho_0 \equiv \frac{B}{2\pi} \tag{3.13}$$

Since the case *ii*) leads to a ground state energy lower than the case *i*) we understand that the choice (3.9) *ii*) is the correct one for the short distance piece of the Coulomb interaction. For $M_{\alpha\beta}(\bar{z}, z')$ slowly varying it is easy to check that spatial variations always increase the energy. We shall explicitely display this feature for the magnetic excitations in the next section.

For the long distance piece of the Coulomb term *i*) still gives rise to a zero contribution to the energy but *ii*) leads to ill defined expressions. Therefore for the long distance part of the Coulomb interaction the ansatz *i*) should be used instead of *ii*).

For the remaining terms in the lagrangian there is only one way to introduce our density matrix ansatz. Thus for constant configurations we finally obtain

$$L = \mu\rho_0 - (\frac{\omega_c}{2} - \frac{gB}{4m})\rho_0 + \frac{e^2\rho_0^2}{2\sqrt{B}}c \tag{3.14}$$

$$c := 4\pi \int_{\frac{\lambda}{\sqrt{B}}}^{\infty} dk e^{-\frac{k^2}{2}}$$

Since the Coulomb term gives a contribution to the ground state energy and we want to have the lowest Landau level of one spin degree of freedom filled up, we have to tune the chemical potential accordingly. This is

$$\mu = (\frac{\omega_c}{2} - \frac{gB}{4m}) - \frac{e^2\rho_0}{2\sqrt{B}}c \tag{3.15}$$

4. Effective action for the lowest lying excitations

Whenever we have a situation of spontaneous symmetry breaking, the Goldstone theorem ensures that there is no gap in the spectrum [13]. Furthermore, the lowest lying excitations are quasiparticles (Goldstone bosons) with well defined transformation properties under the original symmetry group [9]. In particular they can be described by a slowly varying unitary matrix U living in the coset G/H, where G is the symmetry group of the lagrangian and H that of the ground state. In our case $G = SU(2)$ and $H = U(1)$. Under $SU(2)$, U transforms non linearly as follows

$$U(z,\bar{z}) \to gU(z,\bar{z})h^{-1}(g,U(z,\bar{z})) \tag{4.1}$$

where $g \in SU(2)$ is space-time independent whereas $h \in U(1)$ depends on the space-time point through U. We use U to built up our ansatz for the lowest lying excitations. Essentially we specify our density matrix ansatz so that it includes U and it respects the transformation properties of $\rho(z,\bar{z})$, namely

$$\rho(z,\bar{z}) \to g\rho(z,\bar{z})g^{-1} \tag{4.2}$$

These requirements still leave a certain measure of freedom in the ansatz, which is further constrained by demanding the terms with a time derivative in the effective action to be invariant. Up to and including order $1/B$, this reads

$$\begin{aligned}
\hat{\rho} = \frac{B}{2\pi}\int d^2\xi e^{-\frac{B}{2}\bar{\xi}\xi} \ddagger\tilde{U}(\hat{z},\hat{\bar{z}})\ddagger|\xi><\bar{\xi}|\ddagger\tilde{U}^{\dagger}(\hat{z},\hat{\bar{z}})\ddagger \\
\tilde{U}(\hat{z},\hat{\bar{z}}) :=& U(\hat{z},\hat{\bar{z}}) - \frac{1}{B}\Big(\partial_z\partial_{\bar{z}}U(\hat{z},\hat{\bar{z}}) \\
&- \partial_z S\partial_{\bar{z}}U + \partial_{\bar{z}}S\partial_z U \\
&+ \frac{3}{2}\partial_z S\partial_{\bar{z}}SU - \frac{3}{2}\partial_{\bar{z}}S\partial_z SU\Big)
\end{aligned} \tag{4.3}$$

$$S := Up_+U^{\dagger} \qquad , \quad p_+ := \frac{1+\sigma_3}{2}$$

Although the time is not explicitly displayed above, $\hat{\rho}$ may be considered a single time function for all the terms in the lagrangian except for the time derivative term, where the time derivative is to be understood as acting on $\tilde{U}(\hat{z},\hat{\bar{z}})$ only and not on $\tilde{U}^\dagger(\hat{z},\hat{\bar{z}})$. By using the formulae in Appendix A, one can easily see at this order that

$$\begin{aligned}
<\bar{z}|\hat{\rho}|z> = & e^{\frac{B}{2}\bar{z}z}\rho_0\Big(Up_+U^\dagger + \\
& + \frac{1}{B}\big(\partial_z\partial_{\bar{z}}(Up_+U^\dagger) + (\partial_z Up_+\partial_{\bar{z}}U^\dagger - \partial_{\bar{z}}Up_+\partial_z U^\dagger) \\
& - 2Str(\partial_z Up_+\partial_{\bar{z}}U^\dagger - \partial_{\bar{z}}Up_+\partial_z U^\dagger) \\
& + \partial_z Str(U^\dagger\partial_{\bar{z}}Up_+) - \partial_{\bar{z}}Str(U^\dagger\partial_z Up_+) \\
& - 3\partial_z S\partial_{\bar{z}}SS + 3\partial_{\bar{z}}S\partial_z SS\big)\Big)
\end{aligned} \tag{4.4}$$

which enjoys the desired transformation properties (4.2). ¿From (4.4) it follows immediately that the density reads

$$\begin{aligned}
(\rho(z,\bar{z}))_{\alpha\alpha} &= \rho_0\left(1+\frac{2}{B}\Omega\right) \\
\Omega &:= tr(\partial_z Up_+\partial_{\bar{z}}U^\dagger - \partial_{\bar{z}}Up_+\partial_z U^\dagger)
\end{aligned} \tag{4.5}$$

Upon integration over space, Ω gives rise to a topological invariant, the Pontryagin index. Notice that although this term is a total derivative, it is not a total derivative of an $SU(2)$ invariant object, and hence it should not be dropped.

Let us next calculate the contributions of the different terms in the lagrangian to the effective action for U.

Consider first the short distance part of the Coulomb term. By introducing center of mass and relative coordinate we have

$$\begin{aligned}
L_{c,s} = & -\frac{1}{2}\int d^2Zd^2z e^{-B|Z|^2-\frac{B}{4}|z|^2}\psi^\dagger(Z-\frac{z}{2})\psi^\dagger(Z+\frac{z}{2})V_s(z,\bar{z})\psi(\bar{Z}+\frac{\bar{z}}{2})\psi(\bar{Z}-\frac{\bar{z}}{2}) \\
= & \int d^2Zd^2z e^{-\frac{B}{2}|z|^2}V_s(z,\bar{z}) \\
& \Big(e^{-\frac{z}{2}\partial_Z+\frac{\bar{z}}{2}\partial_{\bar{Z}}}e^{-\frac{B}{2}|Z|^2}\rho_{\alpha\beta}(Z,\bar{Z})\Big) \\
& \Big(e^{\frac{z}{2}\partial_Z-\frac{\bar{z}}{2}\partial_{\bar{Z}}}e^{-\frac{B}{2}|Z|^2}\rho_{\beta\alpha}(Z,\bar{Z})\Big)
\end{aligned} \tag{4.6}$$

¿From the last expression and (4.4) it is apparent that a derivative expansion about the center of mass coordinates is well defined. Up to order $1/B$ we obtain

$$L_{c,s} = \frac{1}{2}\int d^2Z \frac{e^2\rho_0^2}{\sqrt{B}} c \left(1 + \frac{1}{B}(\frac{b}{c} - 2) tr\,(\partial_z S \partial_{\bar{z}} S) + \frac{2}{B}\Omega\right) \qquad (4.7).$$

c has already been defined in (3.14), and

$$b := \frac{B^{\frac{3}{2}}}{e^2}\int d^2z V_s(z,\bar{z})|z|^2 e^{-\frac{B}{2}|z|^2}$$

where c and b are smooth dimensionless functions of $\lambda/\sqrt{B}$ only. Notice that this term brings up both the kinetic energy for the spin waves and a topological density.

The long distance part of the Coulomb interaction gives, by just substituting (4.5) in (2.29),

$$L_{c,l} = \int d^2z_1 d^2z_2 \frac{2\rho_0^2}{B^2}\Omega(z_1,\bar{z}_1)V_l(z_1 - z_2, \bar{z}_1 - \bar{z}_2)\Omega(z_2,\bar{z}_2) \qquad (4.8)$$

which corresponds to a non-local interaction between topological densities.

The term in the microscopic lagrangian proportional to the density gives (see (2.27))

$$L_d = (-\frac{e^2\rho_0}{2\sqrt{B}}c - \frac{gB}{4m})\frac{2\rho_0}{B}\Omega(z,\bar{z}) \qquad (4.9)$$

where we have substituted (3.15) for the chemical potential. Notice that the first term in (4.9) cancels precisely against the last term in (4.7). Then, the remaining term that is proportional to the Pontryagin density is also proportional to the g factor. This cancellation was however not observed by the authors of [10], who just quoted the last term in (4.7) as the only term proportional to the Pontryagin index. The remaining term still energetically favours topologically nontrivial configurations, the so called skyrmions, but the effects are expected to be small on account of the smallness of g. The term proportional to the Pontryagin density in (4.7) is of the same order in the derivative expansion as the second term in the same equation. Generically, as we shall see, these two terms emerge together from the short-distance part of the Coulomb interaction and topological term has to be retained if the spin-density is. In this particular instance, the topological

term has cancelled out through the specification of the chemical potential. However, it will be generated at subleading orders, as will be seen in section 5. The topological term, is, of course, odd under parity transformations, but this is acceptable as the external magnetic field has already broken the parity symmetry of the system.

The term with the time derivative gives rise to

$$L_t = \rho_0(A_0 - \frac{1}{2B}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho)$$
$$A_\mu := tr(p_+U^\dagger i\partial_\mu U) \tag{4.10}$$

Notice that both terms are invariant under non-linear $SU(2)$ transformations up to total derivatives, and hence are of topological nature. Moreover, they cannot be written in terms of the magnetization in a local form, which shows the obvious advantage of using U instead of $\vec{m}$ as the basic field in the effective Lagrangian. In fact the second piece in (4.10) is the well known abelian Chern-Simons term . Its appearance in QH ferromagnets has been recently pointed out in [10], and our result agrees with that in this article provided we make the identification $A_\mu = iz^\star\partial_\mu z$. However, the authors of [10] were not able to write it in terms of the basic fields of the effective action only, whereas here it appears naturally in this form. Although this term is $1/B$ suppressed with respect to the first term, requiring invariance of the effective action at this order constrains very much our ansatz (4.3), which is the main reason why we have displayed it here. The first term in (4.10) is very interesting [†]. It appears in the literature in a variety of forms, none of them being a local expression in terms of the basic fields of the action, as (4.10) is. In appendix C we give the precise relation between the different forms that this term can be found in.

Finally the Pauli term gives rise to

$$\frac{gB}{m}\frac{\rho_0}{8}\left(tr(\sigma_3U\sigma_3U^\dagger) - \frac{2}{B}\Omega - \frac{5\Omega}{B}tr(\sigma_3U\sigma_3U^\dagger)\right) \tag{4.11}$$

[†] It also arises in a very different context: the effective action for the low momentum modes of heavy quark anti-quark bound states [14], where it was written in the form (4.10) for the first time.

The third term in (4.11) can be regarded as subleading with respect to the first term and may be dropped. However, the second term, which at first sight also seems subleading, must be retained owing to the cancellation of the leading order term between equations (4.7) and (4.9), as discussed above. In fact this term combines with the second term in (4.9) to give a contribution to the Pontryagin density, which however is proportional to g.

Putting all this together we have up to two space and one time derivative

$$\begin{aligned} S_{eff} = \int dt \Big(\int d^2z \big(\rho_0 A_0 + \frac{1}{2} \frac{e^2\rho_0^2}{\sqrt{B}} c \left(\frac{1}{B} (\frac{b}{c} - 2) tr\, (\partial_z S \partial_{\bar{z}} S) \right) \\ - \frac{3g\rho_0}{4m}\Omega + \frac{gB}{m}\frac{\rho_0}{8} \left(tr(\sigma_3 U \sigma_3 U^\dagger) - 2 \right) \big) \\ + \int d^2z_1 d^2z_2 \frac{2\rho_0^2}{B^2} \Omega(z_1, \bar{z}_1) V_l(z_1 - z_2, \bar{z}_1 - \bar{z}_2) \Omega(z_2, \bar{z}_2) \end{aligned} \tag{4.12}$$

If we take now the cut-off λ such that $p << \lambda << \sqrt{B}$ the coeficients c and b become

$$c \sim \sqrt{2\pi^3} \qquad b \sim c \tag{4.13}$$

and the last term reduces to the Coulomb energy due to the topological densities. In this situation (4.13) is in agreement with [5], except for the topological density term which remained unnoticed there.

5. Leading quantum corrections due to higher Landau levels

We show in this section how quantum corrections due to higher Landau levels can be easily obtained within our formalism. We shall restrict ourselves to the leading corrections in $\hbar$, $e^2\sqrt{B}/\omega_c$ and derivatives.

In appendix B we have obtained the Coulomb term in the Landau level basis (see (B.7)). In order to obtain the exact effective action for the spin waves we should integrate out the fields corresponding to all Landau levels but the lowest. This is of course a formidable task which lies beyond our abilities. Nevertheless, we shall be able to identify the parts of the Coulomb term that are to be retained in order to obtain the leading corrections we are looking for.

The terms containing three or four fields in higher Landau levels only contribute at higher loops (≥ 2) and hence they are supressed by powers of $\hbar$. The terms with exactly two fields in higher Landau levels are to be compared with the terms proportional to the Landau level energies. The size of the former is at least $e^2\sqrt{B}/\omega_c$ smaller than that of the latter and hence we can also neglect them. They would give rise to corrections to the propagator and to loops. We are then left with terms with only one field in the higher L.L. These are,

$$
\begin{aligned}
L_c = &-\frac{1}{2}\int d^2Zd^2z\int\frac{d^2k}{2\pi}\frac{e^2}{|k|}\mathrm{e}^{-i\frac{\bar{k}}{2}z-i\frac{k}{2}\bar{z}-B|Z|^2-\frac{B}{4}|z|^2}\sum_n\frac{1}{\sqrt{n!}}\big[(\frac{k}{\sqrt{2B}})^n\psi_0^\dagger(Z-\frac{z}{2})\psi_0^\dagger(Z+\frac{z}{2})\\
&\times\big(\psi_n(\bar{Z}+\frac{\bar{z}}{2})\psi_0(\bar{Z}-\frac{\bar{z}}{2})+(-1)^n\psi_0(\bar{Z}+\frac{\bar{z}}{2})\psi_n(\bar{Z}-\frac{\bar{z}}{2})\big)\\
&+(-\frac{\bar{k}}{\sqrt{2B}})^n\big(\psi_0^\dagger(\bar{Z}-\frac{\bar{z}}{2})\psi_n^\dagger(\bar{Z}+\frac{\bar{z}}{2})+(-1)^n\psi_n^\dagger(\bar{Z}-\frac{\bar{z}}{2})\psi_0^\dagger(\bar{Z}+\frac{\bar{z}}{2})\big)\\
&\times\psi_0(\bar{Z}+\frac{\bar{z}}{2})\psi_0(\bar{Z}-\frac{\bar{z}}{2})\big] \qquad (5.1).
\end{aligned}
$$

Next, recall that the holomorphic (antiholomorphic) factors must be compensated for at some point by corresponding antiholomorphic (holomorphic) factors. Eventually this has to be realised through derivatives on the the slowly varying fields. Since each derivative goes with a factor $1/\sqrt{B}$, the more important terms are those with a smaller number of unpaired holomorphic (antiholomorphic) factors. In other words, the first Landau level fields are the most important ones, which is very reasonable from a physical point of view. The nth Landau level gives a contribution suppressed by $(p/\sqrt{B})^n$, where p is the typical momentum of the spin waves. Then the sum over n in (5.1) reduces to $n = 1$.

Next we separate, as we did in the previous section, the Coulomb term into short distance and long distance contributions, and organise the fields accordingly. For the short

dictance part we have

$$
\begin{aligned}
L_{c,s} =& \frac{1}{2}\int d^2Z d^2z \int_\lambda \frac{d^2k}{2\pi}\frac{e^2}{|k|} e^{-i\frac{\bar{k}}{2}z - i\frac{k}{2}\bar{z}} e^{-\frac{B}{2}|Z|^2 - \frac{3B}{8}|z|^2} \\
& tr\Big(- e^{B\frac{z\bar{Z}}{4} - B\frac{\bar{z}Z}{4}} \left(e^{\frac{z}{2}\partial_Z - \frac{\bar{z}}{2}\partial_{\bar{Z}}} e^{-\frac{B}{2}|Z|^2}\rho(Z,\bar{Z})\right) \\
& \left(\frac{k}{\sqrt{2B}}\psi_1(\bar{Z}+\frac{\bar{z}}{2})\psi_0^\dagger(Z-\frac{z}{2}) + \frac{\bar{k}}{\sqrt{2B}}\psi_0(\bar{Z}+\frac{\bar{z}}{2})\psi_1^\dagger(Z-\frac{z}{2})\right) \\
& + e^{-B\frac{z\bar{Z}}{4} + B\frac{\bar{z}Z}{4}} \left(e^{-\frac{z}{2}\partial_Z + \frac{\bar{z}}{2}\partial_{\bar{Z}}} e^{-\frac{B}{2}|Z|^2}\rho(Z,\bar{Z})\right) \\
& \left(\frac{k}{\sqrt{2B}}\psi_1(\bar{Z}-\frac{\bar{z}}{2})\psi_0^\dagger(Z+\frac{z}{2}) + \frac{\bar{k}}{\sqrt{2B}}\psi_0(\bar{Z}-\frac{\bar{z}}{2})\psi_1^\dagger(Z+\frac{z}{2})\right)\Big)
\end{aligned}
\tag{5.2}
$$

$$
\sim \frac{ie^2c}{4\pi\sqrt{2}}\int d^2Z e^{-\frac{B}{2}|Z|^2}\left(\psi_0^\dagger(Z)\partial_{\bar{Z}}S\psi_1(\bar{Z}) - \psi_1^\dagger(Z)\partial_Z S\psi_0(\bar{Z})\right)
$$

and for the long distance part

$$
\begin{aligned}
L_{c,l} =& -\int d^2z_1 d^2z_2 e^{-\frac{B}{2}|z_1|^2 - \frac{B}{2}|z_2|^2}\int^\lambda \frac{d^2k}{2\pi}\frac{e^2}{|k|} e^{-i\frac{\bar{k}}{2}(z_1-z_2) - i\frac{k}{2}(\bar{z}_1-\bar{z}_2)} \\
& \left(\psi_0^\dagger(z_2)\psi_0(\bar{z}_2) - \frac{B}{2\pi}e^{\frac{B}{2}|z_2^2|}\right)\left(\frac{k}{\sqrt{2B}}\psi_0^\dagger(z_1)\psi_1(\bar{z}_1) - \frac{\bar{k}}{\sqrt{2B}}\psi_1^\dagger(z_1)\psi_0(\bar{z}_1)\right)
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
\sim & -\frac{1}{\pi}\int d^2z_1 d^2z_2 e^{-\frac{B}{2}|z_1|^2 - \frac{B}{2}|z_2|^2}\int^\lambda \frac{d^2k}{2\pi}\frac{e^2}{|k|} e^{-i\frac{\bar{k}}{2}(z_1-z_2) - i\frac{k}{2}(\bar{z}_1-\bar{z}_2)} \\
& \Omega(z_2,\bar{z}_2)\left(\frac{k}{\sqrt{2B}}\psi_0^\dagger(z_1)\psi_1(\bar{z}_1) - \frac{\bar{k}}{\sqrt{2B}}\psi_1^\dagger(z_1)\psi_0(\bar{z}_1)\right)
\end{aligned}
$$

where we have reinserted the background neutralising charge. Here again the background charge does not contribute to the short distance part but it is essential for obtaining a well defined long distance part.

The terms in the lagrangian which are quadratic in the first Landau level field read

$$
\int d^2z e^{-\frac{B}{2}|z|^2}\psi_1^\dagger(z)(i\partial_0 - \omega_c)\psi_1(\bar{z}) \tag{5.4}
$$

where we have dropped subleading contributions due to the Pauli term. The functional integral over the first Landau level field is now gaussian and can be readily carried out.

This leads to the following extra term in the effective action

$$
\begin{aligned}
L^1_{eff} = & \int d^2z e^{-\frac{B}{2}|z|^2} \\
& \Big(-\frac{2}{\pi}\int d^2z_2 \int^\lambda \frac{d^2k}{2\pi}\frac{e^2}{|k|} e^{-i\frac{\bar{k}}{2}(z_2-z)-i\frac{k}{2}(\bar{z}_2-\bar{z})}\Omega(z_2,\bar{z}_2)\frac{k}{\sqrt{2B}}\psi_0^\dagger(z) \\
& - i\frac{e^2 c}{4\pi\sqrt{2}}\psi_0^\dagger(z)\partial_{\bar{z}}S\Big)\left(-\frac{1}{i\partial_0-\omega_c}\right)\Big(i\frac{e^2c}{4\pi\sqrt{2}}\partial_z S\psi_0(\bar{z}) \\
& + \frac{2}{\pi}\int d^2z_2\int^\lambda \frac{d^2k}{2\pi}\frac{e^2}{|k|}e^{-i\frac{\bar{k}}{2}(z_2-z)-i\frac{k}{2}(\bar{z}_2-\bar{z})}\Omega(z_2,\bar{z}_2)\frac{\bar{k}}{\sqrt{2B}}\psi_0(\bar{z})\Big)
\end{aligned}
\tag{5.5}
$$

Since we consider spin configurations slowly varying in time, we can neglect the time derivatives in the denominators. We then obtain at leading order in derivatives

$$
L^1_{eff} \sim -\frac{Be^4c^2}{64\pi^3\omega_c} tr(S\partial_{\bar{z}}S\partial_z S) \tag{5.6}
$$

This term, which arises from the short distance contributions, amounts to a finite renormalization of the kinetic and topological term respectively. The leading term arising from the long distance contribution, becomes local if we take λ much larger than p, the typical momentum of the spin waves. It reads

$$
\frac{e^4}{\pi\omega_c}\int d^2z\ \Omega^2(z,\bar{z}) \tag{5.7}
$$

This term, however, is subleading with respect to four-derivative terms that arise from the lowest Landau level alone, when we improve our ansatz further. The cross terms in (5.5) involving short and long distance contributions vanish. Let us finally mention that a subleading terms in the long distance contribution of (5.5) gives rise to a surprising non-local three-body interaction between the topological densities which reads

$$
\frac{1}{\pi^3\omega_c}\int d^2z_1 d^2z_2 d^2z \partial_{z_2}\Omega(z_2,\bar{z}_2)V(z_2-z,\bar{z}_2-\bar{z})\Omega(z,\bar{z})V(z-z_1,\bar{z}-\bar{z}_1)\partial_{\bar{z}_1}\Omega(z_1,\bar{z}_1) \tag{5.8}
$$

The bottom line of this section is that higher Landau levels produce no qualitative change to the results obtained upon projection to the L.L.L. : only a finite renormalization of the kinetic energy and the topological density terms is encountered at the lowest order

in derivatives. At higher orders, though, qualitatively new terms like (5.8) arise. It is important to remark that for spin waves with very low momentum p or for very large skyrmions, the corrections due to higher Landau levels that we have just calculated, are the dominant ones. Namely, they go as $e^2\sqrt{B}/\omega_c$ against $p/\sqrt{B}$, which is the order of the corrections to our ansatz. A calculation of the latter may be found in [10].

6. Discussions and Conclusions

In this article, we have addressed the issues of spontaneous symmetry breaking in integral quantum Hall systems, the emergence of magnons as the corresponding Goldstone modes and the existence of charged, topologically nontrivial spin configurations (spin skyrmions) as the lowest-lying charge carriers in the system, when the filling fraction deviates marginally from $\nu = 1$.

The starting point has been the microscopic non-relativistic fermionic action that governs a system of planar electrons in a magnetic field orthogonal to the plane. The Coulomb interaction, which normally plays a subsidiary role in discussions of the integral Hall effect, is included from the beginning, as it is expected that the experimentally measured gap in the spectrum is generated precisely by this term in the case of odd-integer fillings.

Contrary to most other approaches to this problem, we have adopted a second- quantised framework in this article, as S.S.B. is handled with the greatest facility with the machinery of quantum field theory. Consequently, we have avoided overt references to many-body wave functions. The information normally carried by these has been supplied through the specification of the density matrix, describing the ground state, and further on, the lowest excited states of the system. Once the density matrix has been provided, the computation of the physical properties of the system is rendered rather tractable. In our case, we have made a very natural ansatz for the density matrix describing the $\nu = 1$ many-body ground state. The mean energy of the ground state is then computed and we discover that this energy is minimised if the ansatz is adjusted to describe electrons in the

L.L.L., with their spins *spontaneously aligned in a given direction,* even when the Landé g-factor is zero. This is thus an unambiguous demonstration of S.S.B. In fact, within our approach, it is easy to see that it is the Coulomb interaction, projected onto the L.L.L. which is responsible for this spontaneous alignment of the spins. Thus, as a first step, we have established that the ground state of the $\nu = 1$ system is ferromagnetic. This result is quite well-known. However, we believe that our approach possesses the merits of directness and clarity.

Having established the nature of the ground state, we have generalised the ansatz to include the lowest-lying excitations, both neutral and charged. This is done, using a technique quite familiar from high energy physics. The density matrix of the excited state is the unitary transform of that describing the ground state, where the matrices implementing this transformation live in the coset space G/H, where G is the original symmetry group and H the unbroken subgroup. The density matrix has to be covariant under the original symmetry group. In our case, since the projection onto the L.L.L. naturally entails the non-commutativity of the components of the coordinate vectors, the unitary transformation has been carefully defined to ensure this covariance, to a given order in the natural expansion parameter, $\frac{1}{B}$.

What this unitary transformation actually describes is the local rotation of the spins from their fixed direction in the ferromagnetic ground state. Hence, the mean microscopic action, computed with this new density matrix, naturally yields the effective action governing the spin dynamics.

Since the ground state is ferromagnetic, the magnons should, as is well known, possess a quadratic dispersion relation. This in turn means that the time derivative term in the effective Lagrangian should be linear. In fact, we obtain the term with one time derivative in a simple and covariant form, which is equivalent to more conventional forms found in the literature, as we have proved in appendix C. In this appendix, we also show that this term can be cast into the form of a two dimensional Wess-Zumino term. As we have mentioned,

the spin alignment in the ferromagnetic ground state is wrought by the short-distance part of the Coulomb interaction. This correspondingly, would imply that the short-distance part of the Coulomb interaction should yield the spatial gradient term in the effective spin Lagrangian, which measures the energy required to bring the spins out of alignment. This is indeed what emerges from our calculations.

As we have noted, the projection to the L.L.L. leads to a very interesting feature, namely, a local alteration of the charge density is automatically accompanied by a local change in the spin density. Turning the argument around, if one creates a topologically nontrivial spin texture, this configuration would carry charge. We show, very simply and directly, that our ansatz for the excited state leads immediately to a charge density that could yield a nontrivial topological charge with a non-zero Pontryagin index. The O(3) non-linear σ model that describes the magnons is well-known to possess skyrmionic solutions. What is novel in the Hall systems is that these skyrmions are charged. They have an electric charge which equals their topological charge. Moreover, in the effective Lagrangian, we find a term proportional to the topological density, which energetically favours topologically non-trivial configurations. However, this term is also proportional to the g factor and hence is expected to give a small contribution. We obtain an interaction among these charged topological objects mediated by the long-distance part of the Coulomb interaction. It is also worth pointing out that the C-S term that we have obtained depends in a simple way on the basic field U, so that no auxiliary variable is needed, unlike in [10], to build it up.

Most of these above features have been noted in other works, but not all together within a single framework. In [3,5], the time derivative, kinetic and long-range interaction terms were found, but the topological density term was missing, as well as the Hopf term. On the other hand, our results essentially agree with those of [10], except for the coefficient of the Pontryagin density. This is because a crucial cancellation had remained unnoticed by these authors. In fact, in order to obtain all the terms, it is important to separate the

Coulomb interaction between high and low momentum pieces. An important feature of our approach, which we have focussed on in the previous section, is the facility it affords in including the effects of the higher L.L., which have been totally ignored in previous works. In principle, our second-quantised approach incorporates a systematic expansion in the various parameters $\sqrt{B}, \omega_c, e^2\sqrt{B}$. For low energy spin waves, say with energy E, or for skyrmions with a large size, these corrections, $\sim \frac{e^2\sqrt{B}}{\omega_c}$, are more important than those coming from higher order terms in the derivative expansion, $\sim \frac{E}{e^2\sqrt{B}}$. This means that we were able to compute the effects of higher L.L. mixing due to the Coulomb term, rather straightforwardly. These lead to a renormalisation of the parameters of the leading order effective Lagrangian. This is not easily accomplished by the other methods currently available in the literature. This aspect becomes even more important when the electrons are coupled to electromagnetic probes. After all, the skyrmions, as charged objects are touted as the favoured charge carriers in the $\nu = 1$ system and one would like to investigate their electromagnetic responses. The electromagnetic fields would however set up virtual transitions and consequent mixing of the Landau levels. We thus expect our method, which is geared towards including such mixings, to be invaluable in this situation. Work in this direction is currently in progress.

In conclusion, we may summarise the contents of this article as follows. We have used field theoretic techniques to demonstrate that the ground state of the $\nu = 1$ system is ferromagnetic. Furthermore, we have extracted the effective action governing the resultant Goldstone modes, the magnons. In this effective action, we have found several topological terms. The leading term with a single time derivative, the Chern-Simons (Hopf) term and the term proportional to the Pontryagin index. Moreover, we have established a systematic algorithm, whereby contributions to the spin effective action, which arise from L.L. mixing, may be systematically computed. This, we reiterate, is the most significant part of this work.

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Appendix A

In this appendix, we shall elaborate upon some results used extensively in the main text.

a.

$$\langle\bar{\zeta}|\sharp f(\hat{Z},\hat{\bar{Z}})\sharp|\eta\rangle = \mathrm{e}^{\frac{2}{B}\partial_\eta\partial_{\bar{\zeta}}}\langle\bar{\zeta}| : f(\hat{Z},\hat{\bar{Z}}) : |\eta\rangle \qquad (A.1).$$

This is shown as follows:

$$f(\hat{Z},\hat{\bar{Z}}) = \int d\vec{k} f(\vec{k})\mathrm{e}^{i\frac{\bar{k}}{2}\hat{Z}+i\frac{k}{2}\hat{\bar{Z}}} \qquad (A.2).$$

Thus, by definition,

$$\sharp f(\hat{Z},\hat{\bar{Z}})\sharp = \int d\vec{k}\mathrm{e}^{i\frac{\bar{k}}{2}\hat{Z}}\mathrm{e}^{i\frac{k}{2}\hat{\bar{Z}}} f(\vec{k}) \qquad (A.3).$$

Using the result

$$\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^B\mathrm{e}^A\mathrm{e}^{[A,B]}$$

with $A \equiv i\frac{\bar{k}}{2}\hat{Z}$, $B \equiv i\frac{k}{2}\hat{\bar{Z}}$ and $[A,B] = -\frac{\vec{k}^2}{2B}$, we get, from (A.3),

$$\sharp f(\hat{Z},\hat{\bar{Z}})\sharp = \int d\vec{k}\mathrm{e}^{-\frac{\vec{k}^2}{2B}}\mathrm{e}^{i\frac{k}{2}\hat{\bar{Z}}}\mathrm{e}^{i\frac{\bar{k}}{2}\hat{Z}} f(\vec{k}) \qquad (A.4).$$

Thus, sandwiching (A.4) between coherent states, we get (A.1) directly.

b.

Another expression that we repeatedly encounter is:

$$A_{f,g} \equiv \langle\bar{z}|\Big[\int d^2\xi\ \mathrm{e}^{-\frac{B}{2}|\xi|^2} : f(\hat{Z},\hat{\bar{Z}}) : |0,\xi\rangle\langle 0,\bar{\xi}| : g(\hat{Z},\hat{\bar{Z}}) :\Big]|z\rangle \qquad (A.5).$$

Since we know the action of $\hat{Z}$ and $\hat{\bar{Z}}$ on the coherent states, we obtain from (A.5)

$$A_{f,g} = \int d^2\xi\mathrm{e}^{-\frac{B}{2}|\xi|^2+\frac{B}{2}\bar{z}\xi+\frac{B}{2}\bar{\xi}z} f(\xi,\bar{z})g(z,\bar{\xi}) \qquad (A.6).$$

Shifting, $\xi = \eta + z$ and $\bar{\xi} = \bar{\eta} + \bar{z}$, we get

$$A_{f,g} = \mathrm{e}^{\frac{B}{2}|z|^2} \int d^2\eta \mathrm{e}^{-\frac{B}{2}|\eta|^2} f(z+\eta, \bar{z}) g(z, \bar{z}+\bar{\eta}) \qquad (A.7),$$

which may be expanded as

$$A_{f,g} = \mathrm{e}^{\frac{B}{2}|z|^2} [f(z,\bar{z})g(z,\bar{z}) + \frac{2}{B}\partial_z f \partial_{\bar{z}} g + \cdots] \qquad (A.8),$$

where the dots indicate terms of $O(\frac{1}{B^2})$.

Appendix B

In section 5, we have described the contributions to the effective action for the fermionic spin, arising from the mixing of the higher L.L. with the L.L.L. This mixing is brought about by the Coulomb term. In this appendix, we derive an expression for the Coulomb term, projected onto the L.L. basis.

The bare Coulomb term may be expressed as

$$V(\hat{\vec{r}_1} - \hat{\vec{r}_2}) = \int \frac{d\vec{k}}{2\pi} \mathrm{e}^{i\vec{k}\cdot(\hat{\vec{r}_1} - \hat{\vec{r}_2})} \frac{e^2}{|\vec{k}|} \qquad (B.1)$$

where $\vec{r}_i \equiv (x_i, y_i)$.

Thus, the Coulomb Lagrangian is

$$L_c = -\int \frac{d\vec{k}}{4\pi} \frac{e^2}{|\vec{k}|} \sum_{n_1,n_2,n_1',n_2'=0}^{\infty} \int \prod_{i=1}^{2} d^2\xi_i d^2\xi_i' \mathrm{e}^{-\frac{B}{2}(|\xi_i|^2 + |\xi_i'|^2)} W_{n_1,n_1',n_2,n_2'}(\vec{k}) \\ \hat{\psi}^\dagger_{n_1}(\xi_1)\hat{\psi}^\dagger_{n_2}(\xi_2)\hat{\psi}_{n_2'}(\bar{\xi_2'})\hat{\psi}_{n_1'}(\bar{\xi_1'}) \qquad (B.2)$$

where

$$W_{n_1,n_1',n_2,n_2'}(\vec{k}) \equiv \langle n_1, \bar{\xi_1}| \langle n_2, \bar{\xi_2}| \mathrm{e}^{-i\vec{k}\cdot(\hat{\vec{r}_1} - \hat{\vec{r}_2})} |n_2', \xi_2'\rangle |n_1', \xi_1'\rangle \qquad (B.3).$$

Expressing the coordinate operators in terms of $\hat{Z}, \hat{\bar{Z}}, \hat{\Pi}, \hat{\Pi}^\dagger$ we get

$$W_{n_1,n_1',n_2,n_2'}(\vec{k}) = \mathrm{e}^{-i\frac{k}{2}(\bar{\xi_1} - \bar{\xi_2}) - i\frac{\bar{k}}{2}(\xi_1' - \xi_2')} \mathrm{e}^{\frac{B}{2}(\bar{\xi_1}\xi_1' + \bar{\xi_2}\xi_2')} M_{n_1,n_1'}(\vec{k}) M_{n_2,n_2'}(-\vec{k}) \qquad (B.4).$$

Here,

$$M_{n,n'}(\vec{k}) \equiv \langle n|\mathrm{e}^{\frac{k}{2B}\hat{\Pi}}\mathrm{e}^{-\frac{\bar{k}}{2B}\hat{\Pi}^\dagger}|n'\rangle \qquad (B.5),$$

with $k \equiv k_x + ik_y$ and $\bar{k} \equiv k_x - ik_y$.

Upon simplification,

$$M_{n,n'}(\vec{k}) = \frac{1}{\sqrt{n!n'!}}(-\frac{\bar{k}}{\sqrt{2B}})^{n-n'} \sum_{l=max(0,n'-n)}^{\infty} \frac{(n+l)!}{l!(n-n'+l)!}(-\frac{k^2}{2B})^l \qquad (B.6).$$

Using $\mathrm{e}^{\frac{B}{2}(\bar{\xi}_2+\frac{i\bar{k}}{B})\xi_2'} = \langle\bar{\xi}_2 + \frac{i\bar{k}}{B}|\xi_2'\rangle$, we get,

$$\mathrm{e}^{-\frac{B}{2}|\xi_2|^2+i\frac{\bar{k}}{2}\bar{\xi}_2}\int d^2\xi_2'\ \mathrm{e}^{-\frac{B}{2}|\xi_2'|^2}\mathrm{e}^{\frac{B}{2}(\bar{\xi}_2+\frac{i\bar{k}}{B})\xi_2'}\hat{\psi}_{n_2'}(\bar{\xi}_2') = \mathrm{e}^{\frac{k^2}{2B}-\frac{B}{2}|\xi_2|^2+i\frac{\bar{k}}{2}\xi_2+i\frac{k}{2}\bar{\xi}_2}\hat{\psi}_{n_2'}(\bar{\xi}_2).$$

Using this to integrate the primed variables out, we obtain, from (B.2),

$$L_c = -e^2\int\frac{d\vec{k}}{4\pi}\frac{\mathrm{e}^{\frac{k^2}{B}}}{|\vec{k}|}\sum_{n_1,n_2,n_1',n_2'=0}^{\infty} M_{n_1,n_1'}(\vec{k})M_{n_2,n_2'}(-\vec{k})\int\prod_{i=1}^{2}d^2\xi_i\mathrm{e}^{-\frac{B}{2}|\xi_i|^2}\mathrm{e}^{-i\frac{k}{2}(\bar{\xi_1}-\bar{\xi_2})-i\frac{\bar{k}}{2}(\xi_1-\xi_2)}$$
$$\hat{\psi}^\dagger_{n_1}(\xi_1)\hat{\psi}^\dagger_{n_2}(\xi_2)\hat{\psi}_{n_2'}(\bar{\xi_2})\hat{\psi}_{n_1'}(\bar{\xi_1}) \qquad (B.7).$$

This is the Coulomb Lagrangian, projected onto the L.L. basis. The term with all the L.L. indices set to zero is the Coulomb interaction projected onto the L.L.L. and has been used extensively in the earlier sections of the article.

¿From (B.6), it is easily seen that,

$$M_{n,n}(\vec{k}) = M_{n,n}(-\vec{k}) = \frac{1}{n!}\frac{\partial^n}{\partial\rho^n}|_{\rho=-\frac{k^2}{2B}}(\rho^n\mathrm{e}^\rho) = \mathrm{e}^{-\frac{k^2}{2B}} + O(1/B) \qquad (B.8)$$

for all $n \geq 0$. Again, for $n \neq 0$, we have

$$M_{0,n}(\vec{k}) = \frac{1}{\sqrt{n!}}(\frac{k}{\sqrt{2B}})^n\mathrm{e}^{-\frac{k^2}{2B}};\ M_{n,0}(\vec{k}) = \frac{1}{\sqrt{n!}}(\frac{-\bar{k}}{\sqrt{2B}})^n\mathrm{e}^{-\frac{k^2}{2B}} \qquad (B.9).$$

This shows quite clearly that mixing between different Landau levels is suppressed by powers of $\frac{1}{\sqrt{B}}$ and hence, if we look at the mixing of higher L.L. with the L.L.L., it would suffice to consider only $n = 1$.

Appendix C

In this appendix we make contact with various representations of the Goldstone fields found in the literature.

Let us start with $M = U\sigma^3 U^\dagger$ and recall that

$$S = Up_+U^\dagger = \frac{1}{2} + \frac{1}{2}M \qquad , \quad M^\dagger = M \qquad , \quad trM = 0 \qquad , \quad M^2 = 1 \tag{C.1}$$

Then we can write

$$M = \vec{\sigma}\vec{m} \qquad , \quad \vec{m}^2 = 1 \tag{C.2}$$

Since $M \to gMg^{-1}$, $\vec{m}$ transforms in the vector representation of $SU(2)$. In terms of $\vec{m}$, for instance, the kinetic energy term reads

$$4tr(\partial_z S \partial_{\bar{z}} S) = tr(\partial_z M \partial_{\bar{z}} M) = 2\partial_z \vec{m} \partial_{\bar{z}} \vec{m} \tag{C.3}$$

It is also very easy to check that

$$2\Omega = tr(\partial_z U\sigma^3 \partial_{\bar{z}} U^\dagger - \partial_{\bar{z}} U\sigma^3 \partial_z U^\dagger) = -\frac{i}{8}\epsilon^{\alpha\beta} tr(M\partial_\alpha M \partial_\beta M) = \frac{1}{4}\epsilon^{\alpha\beta}\epsilon_{ijk} m^i \partial_\alpha m^j \partial_\beta m^k \tag{C.4}$$

$(\alpha, \beta = x, y)$ which is a more conventional form of the topological density.

Next, consider the term with the time derivative

$$T = 2A_0 = tr(\sigma_3 U^\dagger i\partial_0 U) \tag{C.5}$$

As mentioned before this term cannot be written in terms of S or $\vec{m}$ in a local form. However, we can write it locally in S or $\vec{m}$ if we introduce an extra dimension in the following way. We interpolate smoothly the Goldstone fields $\pi^\alpha(x) \to \pi^\alpha(x, \lambda) \quad , \lambda \in [0, 1]$ in such a way that $\pi^\alpha(x, 1) = \pi^\alpha(x)$ and $\pi^\alpha(x, 0) = 0$. Then it is very easy to check that

$$T = -\frac{i}{4}\int_0^1 d\lambda \epsilon^{\alpha\beta} tr(M\partial_\alpha M \partial_\beta M) = \frac{1}{2}\int_0^1 d\lambda \epsilon^{\alpha\beta}\epsilon_{ijk} m^i \partial_\alpha m^j \partial_\beta m^k \tag{C.6}$$

$(\alpha, \beta = 0, \lambda)$. To our knowledge, the last expresion for T was first written down in [15], whereas both the second last expression, which resembles a two dimensional Wess-Zumino term, and (C.5) are original. T is usually found with a rather different aspect, namely,

$$T = \vec{A}(\vec{m})\partial_0 \vec{m} \qquad , \quad \epsilon_{ikj} \frac{\partial A^k}{\partial m^j} = m^i \tag{C.7}$$

It is simple to check that (C.7) and (C.6) lead to the same equation of motion.